

Multi Solitons of a Bose-Einstein Condensate in a Three-Dimensional Ring

L. Salasnich

*CNR-INFM, Research Unit of Milano University,
Dipartimento di Fisica, Università di Milano,
via Celoria 16, 20133 Milano, Italy
E-mail: salasnich@mi.infm.it*

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A Bose-Einstein Condensate (BEC) made of alkali-metal atoms at ultra-low temperatures is well described by the three-dimensional cubic nonlinear Schrödinger equation, the so-called Gross-Pitaevskii equation (GPE). Here we consider an attractive BEC in a ring and by solving the GPE we predict the existence of bright solitons with single and multi-peaks, showing that they have a limited domain of dynamical stability. We also discuss finite-temperature effects on the transition from the uniform phase to the localized phase.

Key words: Matter waves; Solitons in Bose-Einstein condensates

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Single and multiple bright soliton configurations in a Bose Einstein condensates (BEC) have been observed in two different experiments [1, 2]. These solitons, which can travel for long distances without dispersion, have been theoretically investigated by various authors due to their relevance in nonlinear atom optics [3, 4]. Very interesting appears the not yet experimentally studied case of an attractive BEC in a ring. For this system it has been predicted, on the basis of the one-dimensional (1D) theory, a quantum phase transition from a uniform condensate to a bright soliton [5].

Here we consider an attractive BEC in a 3D ring, taking into account transverse variations of the BEC width. We show that the phase diagram of the system reveals novel and peculiar phenomena: the localized soliton has a limited existence and stability domain and the system supports also multi-peak solitons, which can be dynamically stable.

The time-dependent Gross-Pitaevskii equation (GPE) for a BEC in a trapping potential $U(\mathbf{r})$ is given by

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) - \frac{4\pi\hbar^2 a_s (N-1)}{m} |\psi|^2 \right] \psi = 0$$

where N is the number of condensed atoms and a_s the s-wave scattering length of inter-atomic interaction. The GPE describes N trapped bosons in the same quantum state $\psi(\mathbf{r}, t)$: a pure BEC of N atoms. The GPE is a mean-field Hartree equation and it is accurate for a dilute gas of condensed bosons at ultra-low temperature. The cubic term is obtained by using a contact pseudo-potential $V(\mathbf{r}, \mathbf{r}') = 4\pi\hbar^2 a_s / m \delta^3(\mathbf{r} - \mathbf{r}')$ to model the two-body interaction. If $a_s > 0$ there is repulsion between particles, if $a_s < 0$ there is attraction between particles [6].

We describe an attractive BEC in a cylinder, i.e. under transverse radial harmonic confinement and without axial longitudinal confinement, by using the self-focusing GPE ($a_s < 0$) with ω_\perp the transverse frequency of harmonic confinement. This self-focusing 3D GPE is the Euler-Lagrange equation of the following Lagrangian density

$$\mathcal{L} = \psi^* \left[i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 - \frac{1}{2} (x^2 + x^2) + \frac{1}{2} \Gamma |\psi|^2 \right] \psi$$

where $\Gamma = 4\pi(N-1)|a_s|/a_\perp$ is the scaled inter-atomic strength. Here we use scaled units: energy

in units $\hbar\omega_\perp$, time in units of ω_\perp^{-1} , length in units of $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$. To simplify the 3D problem, let us choose the trial wave function $\psi(\mathbf{r}, t) = f(z, t)\exp[-(x^2 + y^2)/2\sigma(z, t)^2]/\pi\sigma(z, t)^2$. Inserting it into the Lagrangian, integrating over x and y and neglecting the derivatives of $\sigma(z, t)$ one finds an effective Lagrangian density $\tilde{\mathcal{L}}$ which depends on the axial wave function $f(z, t)$ and the transverse width $\sigma(z, t)$. The two Euler-Lagrange equations of the effective Lagrangian $\tilde{\mathcal{L}}$ with respect to f and σ are

$$i\frac{\partial}{\partial t}f = \left[-\frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{2}\left(\frac{1}{\sigma^2} + \sigma^2\right) - \frac{g}{\sigma^2}|f|^2 \right] f$$

$$\sigma^2 = \sqrt{1 - g|f|^2}$$

where $g = \Gamma/(2\pi) = 2(N-1)|a_s|/a_\perp$. For $\sigma = 1$ the first equation becomes the familiar self-focusing 1D GPE, also called cubic nonlinear Schrödinger equation. The equation for σ can be inserted into the equation for f . In this way one gets the time-dependent non-polynomial Schrödinger equation (NPSE) [7].

In [7] we have verified that the NPSE gives results always very close to that of the 3D GPE under transverse harmonic confinement for both positive and negative scattering length a_s . The self-focusing NPSE ($a_s < 0$) admits a shape-invariant solution, called bright soliton, due to the interplay between the dispersive kinetic term and the attractive interaction term. Note that bright solitons of the self-focusing 3D GPE do not exist without transverse confinement. We have found [3, 4] that the bright soliton reduces, in the limit of very strong transverse confinement, namely for $g \ll 1$ (1D regime), to the text-book bright soliton of the self-focusing 1D GPE, whose axial wave function is $f(z) = \sqrt{g/4} \operatorname{sech}(gz/2)$. Contrary to the self-focusing 1D GPE, the bright soliton of the self-focusing NPSE has a limited domain of existence: $0 < g < \frac{4}{3}$. If the strength g exceeds $4/3$ there is the so-called collapse of the BEC. Thus, the transverse dynamics of the BEC induces the instability of the solitonic solution.

As previously stressed, bright solitons have been observed in two experiments: a single soliton in expulsive trap at the Ecole Normale Supérieure (ENS)

of Paris [2], and a soliton train in a harmonic trap at Rice University [1]. We believe that in the next future there will be experiments on attractive BEC in a toroidal trap. In fact, toroidal traps have been recently produced: at Georgia Tech cold atoms in a magnetic ring (diameter of 2 cm) [8] and at Berkeley a BEC in a magnetic ring (diameter of 3 mm) [9].

To model an attractive BEC in a toroidal trap we consider now the self-focusing NPSE where the solution $f(z, t)$ has periodic boundary conditions:

$$f(z + L, t) = f(z, t)$$

where $L = 2\pi R$ is the azimuthal longitudinal length of the BEC and R is the azimuthal radius of the trap. Due to the periodic boundary conditions the stationary self-focusing NPSE admits a uniform solution, given by $f(z) = 1/\sqrt{L}$. One sees immediately that the uniform solution exists only for $g < 2\pi R$. The stationary NPSE admits also nodeless and nodal localized solutions (solitonic solutions). In Fig. 1 we plot the existence diagram of these solitonic solutions in the (R, g) plane. The diagram is obtained by numerically solving the NPSE. The one-peak localized nodeless solution exists between the two solid lines of Fig. 1. Nodeless two-peak localized solution exists between the two dashed lines while the nodal two-peak localized solution exists between the two dot-dashed lines.

We analyze the dynamical stability of the solutions by numerically solving the time-dependent NPSE taking as initial condition the stationary localized solution $f(z)$ with a very weak perturbation. In the left panels of Fig. 2 we plot the density profile $\rho(z) = |f(z)|^2$ of the solitonic solutions. In the right panels we show the time-dependence of the mean squared widths $\langle z^2 \rangle$ for the weakly perturbed solitonic solutions. A periodic oscillation means that the solitonic solution is dynamically stable. In the calculations we choose a ring axial length $L = 15$ and an interaction strength $g = 1$. We have studied the dynamical stability for various initial conditions. The main results are the following: i) the uniform solution is stable only below the lower curve of existence of the one-peak soliton; ii) the one-peak soliton is stable where it exists; iii) the nodal two-

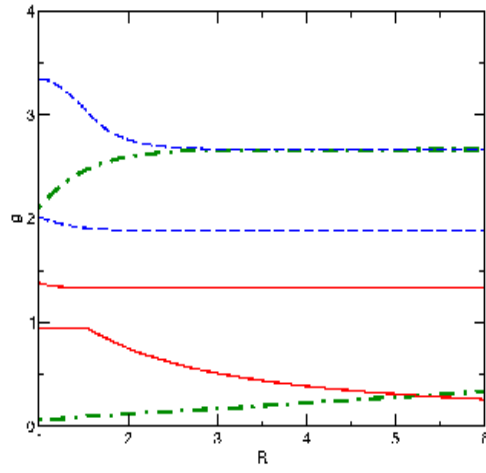


FIG. 1. Existence diagram in the (R, g) plane, where $g = 2N|a_s|/a_\perp$ is the interaction coupling and $R = L/(2\pi)$ is the azimuthal radius of the ring (in units a_\perp). Uniform solution: exists for $g < 2\pi R$. One-peak localized nodeless solution: exists between the two solid lines. Nodeless two-peak localized solution exists between the two dashed lines. The nodal two-peak localized solution exists between the two dot-dashed lines.

peak soliton is stable in the plane (R, g) only below the upper curve of existence of the one-peak soliton; iv) the nodeless two-peak soliton is unstable. Similar results are found for solitonic solutions with a larger number of peaks.

To conclude, we observe that the critical strength $g^* = \pi a_\perp / 2R$ of the transition from the uniform state to the localized state at zero temperature has been determined in [5] by investigating the Bogoliubov energy spectrum of the self-focusing 1D GPE. The inclusion of finite-temperature effects is quite simple if one consider a non-interacting thermal cloud. In this way we find that the critical temperature T^* of transition from the uniform to the localized state is given by

$$\frac{k_B T^*}{\hbar \omega_\perp} = \frac{a_\perp^2}{4R^2} \left(N - \frac{\pi a_\perp^2}{4R|a_s|} \right)$$

where k_B is the Boltzmann constant, and using not-scaled quantities. Obviously if $N < \pi a_\perp^2 / (4R|a_s|)$ the system remains uniform at all temperatures.

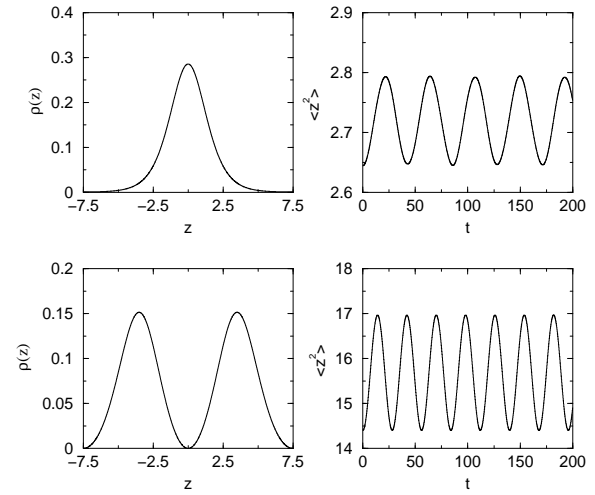


FIG. 2. Left panels: density profile $\rho(z)$ of the solitonic solutions. Right panels: time-dependence of the mean squared widths $\langle z^2 \rangle$ for the weakly perturbed solitonic solutions. Ring axial length: $L = 15$. Interaction strength: $g = 1$.

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